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1 Introduction

In applied science and engineering, polynomials computation has an essential role, since many problems can be represented as operations on polynomials. While some of the basic operations on polynomials seem straightforward and their solutions can be obtained easily, polynomials division (deconvolution) is one of the computational problems. Even if one polynomial is an exact divisor of the other, the result would not be in a polynomial form.

In practical applications, this problem becomes more obvious where the coefficients of the input polynomials are usually expected to have a degree of noise. This noise is due to the rounding off in the previous computations. As a result, the input polynomials being turned into inexact polynomials. Therefore, deconvolving such polynomials will most likely be non trivial.

Over past decades, number of studies in computer algebra have been carried out to investigate such problem. Consequently, the interrelationship between fundamental computations with polynomials and rational functions and computations with structured matrices has been demonstrated. Moreover, it has been shown that structured matrices works as a bridge between computations with polynomials and numerical matrix computations.

Various proposed methods on structured matrices involving deep mathematical tools were found to solve several problems in these fields. In that sense, to work more sensibly with inexact polynomials, each polynomial problem should be translated into the terms of a structured matrix.

This project focuses on solving the problem of two inexact polynomials deconvolution when one polynomial is an exact divisor of the other. It will benefit from the application of structured matrices in polynomials computations. It will investigate the use of the Toeplitz matrix as a structure matrix to represent the input data and formulate the problem of deconvolution as a structured total least squares problem.

1.1 Aim

This project aims to consider the application of a structure preserving method to address the problem of deconvolving two divisible polynomials, when the noise is present in either one polynomial or in both. The following example illustrates the main goal of this work.

Consider the exact polynomials $f(x)$ and $g(x)$ where g is an exact divisor of f:

$$f(x) = (x-1)(x+3) \text{ and } g(x) = (x-1),$$

therefore f/g can be computed easily which is $f/g = (x+3)$.

However, in reality where the coefficients of f and g are affected by the noise which makes:

$$\widehat{f} = f(x) + \triangle f = (x - 1.02)(x + 3.01)$$

$$\text{and } \widehat{g} = g(x) + \triangle g = (x - 1.002),$$

Thus, the result of $\hat{f}/\hat{g}$ will be a rational function.
According to the structured total least norm (STLN) method technique, correcting the noise with the minimal perturbation on $f(x)$ and $g(x)$ leads to a solvable equation. In that sense, the work will focus on computing the solution h of:

$$h = \frac{\widehat{f}+s}{\hat{g}+z},$$

in a polynomial form satisfying the constrain that s and z are in the possible minimization. The strategy of the work is mainly divided into three stages.The first stage is to formulate the polynomials deconvolution into a structured matrix-vector multiplication using a Toeplitz matrix in particular.

Then, in the next stage, the structured total least norm (STLN) method will be applied to solve the resulting least square equality problem. After that, the method will be implemented in a MATLAB program and number of experiments will be conducted to investigate to what extent the (STLN) method will overcome the problem.

1.2 Dissertation Structure

The remaining part of this dissertation is structured as follows:

• Chapter 2: Mathematical Background: This chapter introduces clear descriptions and definitions for some basic mathematical concepts on polynomials and matrices.

• Chapter 3: Literature Review: This chapter presents the proposed approaches in the previous studies to solve the least squares problem. It critically evaluates each approach and outlines the nominated method that is used in the next chapter. In addition, it outlines and concludes some results of previous works.

• Chapter 4: Methodology: This chapter analyzes the main problem of inexact polynomials deconvolution. It gives a clear description on how it can be formulated into a least squares equality (LSE) problem. It summarizes the STLN based algorithm that will be implemented

•Chapter 5: Results and Discussion: This chapter evaluates the proposed algorithm. It presents the results of a number of experiments that have been carried out on a developed MATLAB program. It critically evaluates the method performance and the accuracy of the computational results.

- Chapter 6: Conclusion and Future Work: This chapter concludes the final work with some suggestions for future work.

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2 Mathematical Background

2.1 Introduction

This chapter presents some basic mathematical concepts in polynomials that are necessary to declare and used in the later chapters. In addition to some matrices concepts since each polynomials operation can be translated into a matrix system.

This translation helps to address the ill posed operations in polynomials computations such as polynomials deconvolution which can be turned and formulated into least squares problem.

The chapter firstly, defines general polynomials aspects in the first section then the followed section focuses on the structured matrices. Then, mathematical operations on polynomials such as convolution and deconvolution are explained in section 2.4. The remainder of the chapter illustrates the way of using matrices to solve linear system equations.

2.2 Polynomials

2.2.1 Approximate (inexact) polynomials

In real applications, the polynomial's coefficients have some error added to them. Thus, the polynomial becomes inexact polynomial or approximate polynomial. In practical :

$$\hat{f} = f + \triangle f. \tag{2.1}$$

2.2.2 Polynomial coefficients norms

For each polynomial: $f(x) = \sum_{k=0}^{n} a_k x^k$ and $a = \begin{bmatrix} a_0 & a_1 & \ldots & a_n \end{bmatrix}$ is a vector of its coefficients, it has several classes of norms, which can be defined as:

$$||a||_p = (\sum_{k=0}^{n} |a_i|^p)^{1/p}$$

In this report the $||a||$ will be used as a standard for $||a||_2$.

2.3 Structured Matrix

2.3.1 Toeplitz Matrix

In mathematics, a Toeplitz is a matrix in which each descending diagonal from left to right is constant. The Toeplitz matrix from a given vector $a = \begin{bmatrix} a_0 & a_1 & a_2 \end{bmatrix}$:

$$T(a) = \begin{bmatrix} a_0 & & \\ a_1 & a_0 & \\ a_2 & a_1 & a_0 \\ & a_2 & a_1 \\ & & a_2 \end{bmatrix}$$

2.4 Convolution and Deconvolution

2.4.1 Convolution

Convolution is a mathematical operation that takes two functions f and g to produce a third function h which is a modified version of the two input functions. Algebraically, convolution is equivalent to polynomial multiplications.

2.4.2 Deconvolution

It is a non-trivial operation which is equivalent to polynomials division. If the ratio $f(x)/g(x)$ is a polynomial, a random noise in $f(x)$ and/or $g(x)$ makes it a rational function. Therefore the deconvolution of two inexact polynomials is an ill-posed problem.

2.5 Representing Linear Algebraic Equations in Matrices

It has been shown that matrices provide a concise notation for representing and solving linear equations. For example:

$$\begin{cases} a_1x_1 + b_1x_2 + c_1x_3 = d_1 \\ a_2x_1 + b_2x_2 + c_2x_3 = d_2 \\ a_3x_1 + b_3x_2 + c_3x_3 = d_3 \end{cases} \tag{2.2}$$

The above (1) system of equations can be represented and solved by matrices as follows:

$$Ax = b \tag{2.3}$$

Where A is the matrix of coefficients, b is the column vector of constants and x is the column vector of unknowns :

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$b^T = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}$

$x^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$

Then to solve equation (2):

$x = A^{-1}b$

Therefore, the value of x has been determined. In this work, we will use the Moore–Penrose pseudo inverse.

2.6 Summary

In conclusion, dealing with operations on polynomials requires dealing with matrices. The chapter declared the necessary mathematical concepts and notations related to polynomials together with matrices. A clear explanation of how to represent equations of polynomials as operation on matrices has been provided in the last section of this chapter.

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3 Literature review

3.1 Introduction

This chapter evaluates the different proposed methods taken to address the structured total least squares approach problem. The chapter starts by explaining the least squares problem, discussing how the problem can be extended to the different approaches.

Then, it focuses deeply on the structured Total Least Squares approach, outlining some methods developed to address such problems. Many problems in various areas can be formulated as a structured Total Least Squares problem, for example, system identification, computer algebra, and speech and sound processing. The chapter concludes the computational results of the previous works that have been done an published.

3.2 Least squares problems

The Least Squares problem (LS) involves finding x that satisfies the following minimization:

$$min_x||Ax - b|| \tag{3.1}$$

in order to solve an over determined linear equation: $Ax \approx b$.Many proposed methods to solve such problems allow the noise to be added into vector b only and assuming that the given matrix A is known without error.

However, by allowing the possibility of error in the elements of data matrix A ,we can obtain more accurate solutions for the entire equations. This extension for the LS problem known as total least square (TLS problem)can be stated as finding the vector x with the minimization:

$$min_{||\triangle A||,||\triangle b||}||A + \triangle A)x - (b + \triangle b)|| \tag{3.2}$$

Golub and Van Loan(1980) introduced the basic least squares problem (3.1) along with the solution via singular value decomposition in their paper. Years later, this solution had been generalized to overcome the problem of multivariable and non generic cases.

Finally, the extension of TLS is the structure total least squares (STLS) which satisfies the same minimization needed in TLS while preserving the structure of matrix A. This additional constraint plays an important role in many computer science applications such as signal processing, system identification and system response prediction. A solution to this problem will allow us to form the output polynomials without the concerns of matrix A structure.

De Moor (1994) has mentioned various applications of the structured total least squares problem, with an exception of the numerical solution with the help of a Newton-type optimization method on the constrained total least squares problem.

He has also provided an outline regarding a new framework that would be helpful in deriving numerical methods and analytical properties. He has based his approach on the Lagrange multiplier, which yields equivalent problem termed as Reimannian singular value decomposition.

3.3 Structured Total Least Squares problem

To date, various methods have been developed and introduced to solve STLS problems. Van Huffel & Lemmerling (2002) described three different approaches, which are the Constrained Total Least Square algorithm (CTLS) proposed by Abatzoglou and Mendel in 1987, the Riemannian Singular Value Decomposition (RiSVD) algorithm by De Moor (1993, 1994) and the Structure Total Least Norm (STLN) algorithm by Van Huffel et al. (1996).

While all of these approaches were developed to satisfy the minimization in 3.2, each approach has its own field of applications. The most significant difference that distinguishes the last method (STLN) and makes it straightforward, is that it basically starts from the exact formula of the problem, while the others derive an equivalent formulation for which each algorithm, then develop it in a quite different series of steps.

Lemmerling et al. (1996) declared CTLS approach with the number of different methods that have been used. This paper has proven the convergence of the value obtained by using such an approach with the value obtained in solving TLS problem with other approaches.

However, the convergence rate depends totally on which method has been selected to solve the CTLS approach (Lemmerling et al., 1996). Abatzoglou et al. (1991) identified several advantages of applying the CTLS technique in Harmonic super-resolution problems. Indeed, the CTLS is a useful tool in signal processing problems where the known error present in the data matrix is algebraically associated, and there must be a solution for that equation.

Besides these contributions, a lot of research still needs to be performed into how to improve the performance of that approach.

According to De moor (1994), the STLS problem is equivalent to non-linear singular value decomposition. Then using this technique with one of the proposed methods will result in producing the solution for the STLS problem .

It is evident the efficiency of such an approach to solve the noisy realization problem. Furthermore, the study that has been done by (Fierro and Jiang ,2005) confirmed the reliability of the RiSVD approach in information retrieval applications. However, according to (Van Huffel et al.,1996) the result obtained by such an approach does not guarantee the preservation of the matrix structure, which is a requirement of the optimal solutions for STLS problems.

Rosen et al. (1996) proposed structured total least norm which is another algorithm for calculating structured total least squares solution. This method has been applied to solve a range of problems in various applications such as system identification, speech and audio processing and computer algebra.

The efficiency and robustness of this algorithm in addressing the structured total least squares problems have been proven theoretically and practically especially when the error can occur in the input data.

The choice of STLN approach is supported by the accurate results reported in many published papers and researches ;Winkler and Allan (2008) and (Van Huffel et al., 1996). STLN proved its efficient performance in many approximate polynomials and structured matrices applications where the STLS problems arises.

3.4 Overview of Structured Matrices Applications on polynomials

In light of the previous section, STLN has been used in many different applications. For example, noise realization, image reconstruction, system identification and signal processing. This section highlights some applications.

Winkler and Allan (2008) have developed in their work a method to compute the greatest common divisor (GCD) of two inexact polynomials. The main problem has been formulated into STLN as follow:

$$min||z|| \text{ with } (A_k + E_k)x = b_k + h_k$$

for some vector x,where the perturbation matrix $[h_k, E_k]$ has the same structure of the Sylvester matrix $[b_k, A_k]$ and z is the correction vector.Thus, they was aiming to find x that satisfied the minimum perturbation in both input polynomials.After the STLN method has been examined, an accurate results has been obtained from different test cases without needing to large number of iterations.

Apart of computer algebra, the STLN method also has been used in engineering applications such as signal processing, linear prediction and noise realization problems. More precisely, the Hankel structure has been used in noisy realization problem. According to (Moor B., 1994) for a given exact data $a \in \mathbb{R}^{p+q-1}$by the perturbed $b \in \mathbb{R}^{p+q-1}$, the noise can be realized by solving the following minimization:

$$\sum_{i=1}^{p+q-1}(a_i - b_i)^2 \text{ subject to } By = 0, y^t y = 1,$$

where B is $p \times q$ Hankel matrix constructed from the elements of b.

Markovsky and Huffel (2007) reviewed the total least squares methods in their paper in deep discussion and a comprehensive explanation.They mentioned different applications that used different structured matrices associated with their STLS formulas. Conclusion of their work and the above, it is evident that STLS solutions were the optimal solutions for many problems when its structured appropriately.

3.5 Summary

A considerable amount of literature in least squares problem has been researched. Least squares problem can be extended into total least squares(TLS) and structured total least squares (STLS) approaches depending on the limitation of the problem.

Each approach has its own applications and constrains. In computer algebra, polynomials computations can be turned into STLS problem.

Structured matrices have been studied closely for a long time in somewhat different fields, such as mathematics, computer science and engineering. Number of papers that summarized these studies of the structured matrices have been reviewed in this work.

In summary, STLN that is written by (Rosen et al. ,1996) is the most appropriate method to be used in the deconvolution problem. It will be adapted using appropriate structure (Toeplitz structure). A full description of the proposed method considered in the following chapter.

4 Methodology

4.1 Introduction

This chapter explains the first two stages that have been clarified in the first chapter of this dissertation. It illustrates the problem of deconvolving two inexact polynomials and how it can be formalized into a least square problem.
Then, it provides a suggested method to construct a Toeplitz matrix that will be used to represent the input polynomials coefficients. Next, it describes the method of structured total least norm(STLN) for the solution of the deconvolution problem. Furthermore, it outlines some techniques that are used while developing the MATLAB program.

4.2 Toeplitz matrix-vector Multiplication

Suppose that we have two polynomials $f(x)$ and $g(x)$ of degrees m and n respectively, as follow :

$$f(x) = \textstyle\sum_{i=0}^{m} a_i x^{m-i} \text{ and } g(x) = \sum_{i=0}^{n} b_i x^{n-i}$$

and thus the polynomial

$$h(x) = f(x)/g(x) \tag{4.1}$$

is of degree $(m-n)$,

$$h(x) = \sum_{i=0}^{m-n} h_i x^{m-n} \tag{4.2}$$

Then (4.1) can be written in a matrix-vector multiplication form using a Toeplitz structure as

$$T(g)h = f \tag{4.3}$$

where $T(g) \in \mathbb{R}^{(m+1)\times(m-n+1)}$, $h \in \mathbb{R}^{(m-n+1)}$ and $f \in \mathbb{R}^{(m+1)}$ are coefficients vectors of polynomials $h(x)$ and $f(x)$respectively.

$$T(g) = \begin{bmatrix} b_0 & & & \\ b_1 & b_0 & & \\ \vdots & b_1 & \ddots & \\ & \vdots & \ddots & b_0 \\ b_n & & \ddots & b_1 \\ & b_n & & \vdots \\ & & & b_n \end{bmatrix}, \; h = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{m-n} \end{bmatrix} \text{ and } f = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

The deconvolution problem requires calculating h when f and g are given, which implies finding the least squares solution of (4.3).

4.3 Solving the Least Squares problem

4.3.1 The Moore–Penrose pseudoinverse

The simplest way to find the least squares solution is by using the pseudo inverse as followss:

$$h = T(g)^{\dagger} f, \tag{4.4}$$

This way will be used first to calculate the initial value of h.

4.3.2 The Structured Total Least Norm method

In order to apply the structure preserving method to find the solution for (4.3), it is required that the coefficients of of the input polynomials be perturbed slightly. Therefore, the coefficients of $h(x)$ can be calculated more accurately. Thus, the equation (4.3) will be written as follows

$$(T(g) + E(z))h = f + s, \tag{4.5}$$

where $E(z) \in \mathbb{R}^{(m+)\times(m-n+1)}$ has the same structure of $T(g)$, and the vector $s \in \mathbb{R}^{(m+1)}$ is the correction vector for the polynomial f.

$$E(z) = \begin{bmatrix} z_0 & & & \\ z_1 & z_0 & & \\ \vdots & z_1 & \ddots & \\ & \vdots & \ddots & z_0 \\ z_n & & \ddots & z_1 \\ & z_n & & \vdots \\ & & & z_n \end{bmatrix} \text{ and } s = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{bmatrix}.$$

Thus, the vectors z and s need to be computed. Supposing that the residual r using an approximate solution of (4.5) is

$$r = r(s, z) = (f + s) - (T(g) + E(z))h, \tag{4.6}$$

then

$$\begin{aligned} r(s + \delta s, z + \delta z) \;&=\; (f + (s + \delta s)) - (T(g) + E(z + \delta z))(h + \delta h) \\ &= r(s, z) + \delta s - (T(g) + E(z))\delta h - (\delta(E(z))h, \end{aligned}$$

where

$$\delta E(z) = \sum_{i=0}^{n} \frac{\partial E}{\partial z_{zi}} \delta z_i. \tag{4.7}$$

There exists a matrix $Y(h) \in \mathbb{R}^{(m+1)\times(n+1)}$ that satisfies the following equation:

$$E(z)h = Y(h)z, \tag{4.8}$$

So, we can substitute $(\delta(E(z))h$ in $r(s + \delta s, z + \delta z)$ equation with $Y(h)\delta z$, That leads to change r to:

$$\begin{aligned} r(s + \delta s, z + \delta z) \;&=\; (f + (s + \delta s)) - (T(g) + E(z + \delta z))(h + \delta h) \\ &= r(s, z) + \delta s - (T(g) + E(z))\delta h - (Y(h))\delta z, \end{aligned}$$

In order to solve (4.3) using the Newton-Raphson methods, it implies an iterative solution for the residual

$$\begin{bmatrix} Y & (T + E) & I_{m+1} \end{bmatrix} \begin{bmatrix} \delta z \\ \delta h \\ \delta s \end{bmatrix} = r, \tag{4.9}$$

Hence, it is required to follow the minimization of

$$|| \begin{bmatrix} \delta z & \delta h & \delta s \end{bmatrix} ||, \tag{4.10}$$

Subject to

$$\begin{bmatrix} Y & (T+E) & I_{m+1} \end{bmatrix} \begin{bmatrix} \delta z \\ \delta h \\ \delta s \end{bmatrix} = r, \tag{4.11}$$

That leads to least squares equality problem, which can be solved by the QR decomposition technique. Suppose that

$$F = I_{2m+3,}, \quad G = \begin{bmatrix} Y & (T+E) & I_{m+1} \end{bmatrix}, \quad y = \begin{bmatrix} \delta z \\ \delta h \\ \delta s \end{bmatrix},$$

$$S = \begin{bmatrix} -z_i \\ -(h_i - h_0) \\ -s_i \end{bmatrix} \text{ and } t = r_i,$$

Therefore, this works considers the following LSE problem:

$min_y ||Fy - S||$ Subject to $Gy = t$.

That means we need to overcome the noise that turn the polynomial f/g to a rational function. STLN method would correct the noise with minimizing the perturbation as much as possible.

The following algorithm is generated base on STLN. Since the input of the algorithm is inexact polynomials, a random noise with ratio μ will be added to f and g firstly. The stop condition for the iterative method is when the total norm error in the computed solution $\leq 10^{-12}$ or after 100 iterations. The reason for choosing this number is that no improvement will be noticed when the TN error becomes less than 10^{-12}.

The method denotes to the corrections added to f and g with z and s respectively. The values of these correction vectors initialized by zeros. Taking into account that z has the same structure of $T(g)$. As, it is clear in the algorithm, powerful mathematical techniques will be used such as QR factorization.

Algorithm 1 Deconvolution using QR decomposition.

Input: Inexact polynomials $f(x)$ and $g(x)$.
Output: The polynomial $h(x) = f(x)/g(x)$.
Begin

1. Set $z_0 = 0$, $s_0 = 0$ and calculate $h_0$from (4.4)
2. Repeat

- Compute the QR decomposition of G^T,

$$G^T = QR = Q \begin{bmatrix} R_1 \\ 0 \end{bmatrix}. \tag{4.12}$$

- Set $w_1 = R_1^{-T} \in \mathbb{R}^{(m-n+1)}$.
- Partition FQ as

$$FQ = \begin{bmatrix} F_1 & F_2 \end{bmatrix}, \tag{4.13}$$

where $F_1 \in \mathbb{R}^{(2m+3)\times(m+1)}$ and $F_2 \in \mathbb{R}^{(2m+3)\times(m+2)}$.

- Compute

$$w_2 = F_2^{\dagger}(S - F_1 w_1) \in \mathbb{R}^{(m+2)}. \tag{4.14}$$

- Compute the solution

$$y = Q \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{4.15}$$

- Set $z := z + \delta z$, $h := h + \delta h$ and $s := s + \delta s$.
- Update $E(z)$ and $Y(h)$.
- Update G, S and t, then compute the residual $r(z)$ from (4.7).

Until$\frac{||r(z)||}{||f+s||} \leq 10^{-12}$.

End

4.3.3 Highlight on MATLAB code

Here are functions that are used while developing the MATLAB program. The program takes the input polynomials in roots form. So, it is necessary to generate the polynomial using the roots before starting the iterative method. The following function is written to accomplish this task.

```
function [p]=CreatePolynomial(a)
[row col]=size(a);
p=1;
for i=1:row
    C=[ 1 -(a(i,1))];
    for j=1:a(i,2)
        p=conv(p,C);
    end
end
end
```

Moreover, the Toeplitz matrix of the vector g, should be constructed properly. The function Toeplitz(g,m) is developed to achieve this structure.

```
function T = Toeplitz(g,m)
n=length(g)-1; % the degree of g
T=zeros(m+1,m-n+1);
for k=1:1:m-n+1
    for l=k:1:n+k
        T(l,k)=g(l-k+1);
    end
end
end
```

In order to create the matrix $Y(h)$ using the vector h which satisfies $E(z)h = Y(h)z$, the following MATLB function has been written.

```
function Y=createY(h,m,n)
Y=zeros(m+1,n+1);
for k=1:1:n+1
    for l=k:1:m-n+k
        Y(l,k)=h(l-k+1);
    end
end
end
```

4.4 Summary

In conclusion, the polynomials deconvolution problem has been analyzed and how it can be transformed into convolution with a Toeplitz matrix form has been clarified.

It has been shown how it leads to least squares equality (LSE) problem. An exact solution can be obtained with minimum perturbation by the (STLN) based algorithm which is explained in this chapter.

The proposed algorithm has been implemented in a MATLAB program. In order to validate the method in calculating an exact solution for the deconvolution problem, a series of experiments on the developed software are carried out.

The computational results are summarized and evaluated in the remaining parts of the dissertation.

5 Results and Discussion

5.1 Introduction

To explore the efficiency of STLN method in solving the deconvolution problem, a MATLAB program has been developed based on the methods proposed in chapter 4. This chapter carries out number of experiments and discusses the produced results.

It starts by explaining the test procedure and the properties that need to be investigated. Then the results will be displayed in appropriate format. Finally, it discusses the accuracy of the results in each experiment.

5.2 Test Procedure and Experiments:

5.2.1 Test Procedure

The test procedure is mainly based on performing number of experiments. Each experiment is run to examine a specific criterion of STLN and test its performance under determined input data test case .The main criteria need to be investigated are summarized in the following points:

- The accuracy of calculating the coefficients of the output polynomial h in a correct degree $(m-n)$.
- The ability to overcome the massive degree of the noise in the input data (when μ is too large).
- The ability to maintain its efficiency when the polynomial coefficients vary in their magnitude i.e. the distance between the polynomial roots.
- The robustness and speed of such algorithm when the input polynomials with high degrees (high roots multiplicities) .

In the test we will use the total norm error (TN), which is calculated in each iteration by $\frac{||r(z)||}{||f+s||}$, ans the number of iterations needed to get the answer as a measurement. Also, in some test cases the results will be compared with the simple least squares (LS) solutions.

5.2.2 Experiment 1

The main purpose of this experiment is to make sure that the output h of the algorithm is generated in a polynomial form with the correct degree $(m-n)$.

Input:

$$f(x) = (x-2)^2(x+1)^3(x-7)$$
$$g(x) = (x-2)(x+1) \tag{5.1}$$

.

and the noise to signal ratio μ set to 10^{-4}.
with regard to μ,the input exact polynomials will be perturbed and become:

$$f(x) = 1.0001x^6 - 8.0011x^5 + 2.0004x^4 + 36.0025x^3 + 1.0002x^2 - 52.0033x - 28.0034. \tag{5.2}$$

$$g(x) = 1.0002x^2 - 1.0001x - 2.0001. \tag{5.3}$$

Output:

$$h = 1.0001x^4 - 6.9998x^3 - 2.9988x^2 + 19.0002x + 14.0012. \tag{5.4}$$

The above result is obtained after only 2 iterations with the total norm error = $6.1552e^{-16}$. Now, it is clear that the iterative algorithm STLN works properly and efficiently to produce the result of deconvolving two inexact polynomials in a polynomial form and in a reasonable time. Moreover, the resulting polynomial is in the expected degree $(m-n)$.

5.2.3 Experiment 2

This experiment is performed to evaluate the STLN on the second criterion mentioned in section (5.2.1). It compares the STLN based algorithm solutions with the simple least squares solutions using the total norm error as a measure.The values of f and g in experiment1 are repeated and the solutions are computed at 10 values of μ. It started at 10^{-12} then gradually increased till 10^{-2}. The total norm error (TN) is calculated every time step for each approach. Then, the resulting values of TN errors are plotted.

Input:

Consider the same input data $f(x)$ and $g(x)$ given in experiment 1.

Output:

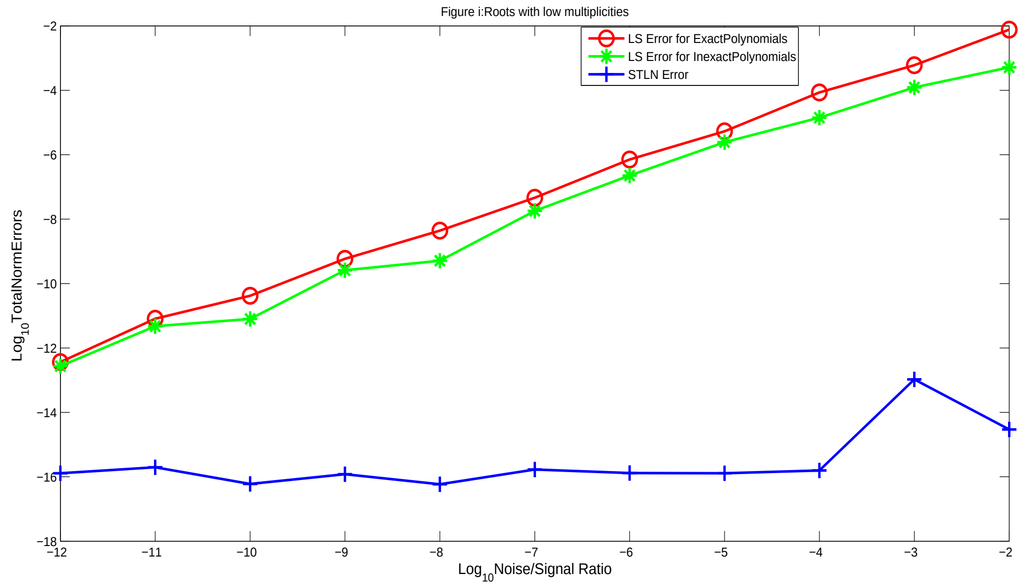

Figure 1: The effect of μon the total norm errors.

According to figure 1, the TN errors for STLN method which are shown in the lower curve, are always much less than LS solutions. For instance, when $\mu = 10^{-9}$, the STLN based TN error $=176e^{-17}$while LS TN error$=435e^{-12}$.

Furthermore, when μ is increased, STLN method persists this massive noise and continues minimizing the TN as much as possible. In STLN practical results, at $\mu = 10^{-11}$ the TN error$=188e^{-17}$ and as μ increased to 10^{-2},the TN error $= 291e^{-13}$. That means the huge increase in μ did not cause a big loss in STLN's efficiency as happened in LS solutions.

It is clear form this experiment that STLN always has the ability to provide the optimal solutions whatever the noise in the input data.

5.2.4 Experiments 3 and 4

This experiments consider the divergence of the input polynomials roots. Each experiment is run to handle a specific case of polynomial roots. The test case in Experiment 3 is when the roots are extremely closed while the distant roots held in experiment 4 The outputs of the experiments are grouped together and plotted in the same figure below.

Experiment 3 input:

$f(x) = (x - 2e^{-3})^2(x - 6e^{-3})^3(x - 1.20)$

$g(x) = (x - 2e^{-3})^2(x - 1.20)$

Experiment 4 input:

$f(x) = (x - 10^{-3})^4(x-2)^7(x+20)^3(x-100)^2.$
$g(x) = (x - 10^{-3})^2(x-2)^1(x+20)^1.$

Output:

The computed results have been plotted against the iterations.

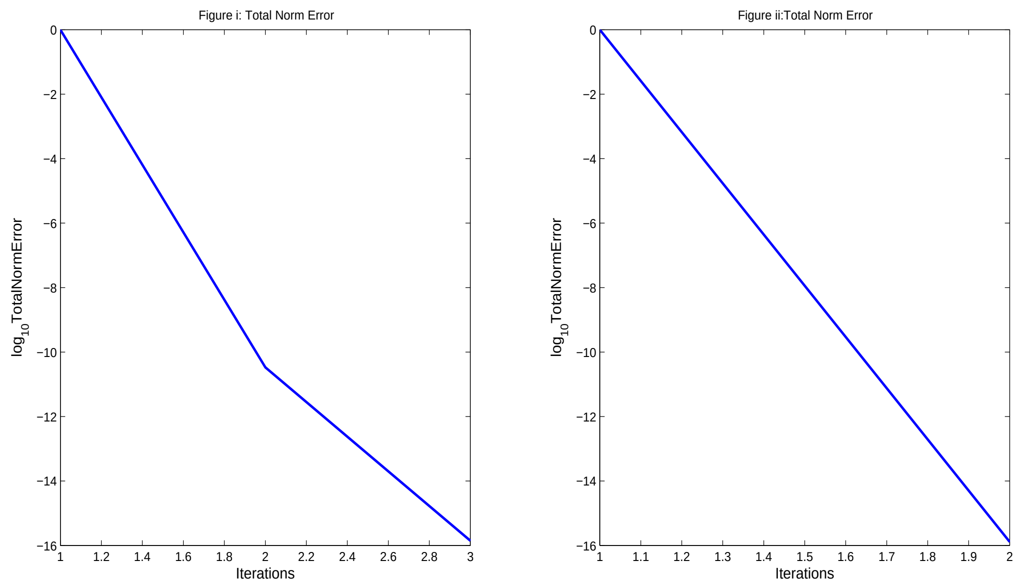

Figure 2: The effect of polynomials roots on STLN solutions at $\mu = 10^{-4}$.
Figure i: when the roots are nearby.
Figure ii:when the roots are distant.

Figure 2 shows how the STLN behaves at two cases of polynomials roots. As appears in figure 2 i, the TN error in the solution is minimized very quickly each time step. Also, the same behavior is repeated in figure 2ii when the roots are distant. In addition, this desired minimization is obtained after no more than two iterations in all different cases.

So, STLN satisfied the required minimization in a reasonable time and regardless of the roots status. The best solution obtained was with TN error $e^{-16}$when the roots are distant.

5.2.5 Experiment 5

The goal of this experiment to explore the last criterion mentioned in section (5.2.1). It is basically made to investigate the robustness when the multiplicities of the roots are very high. It involved running the program with 7 different values of (m, n) at $\mu = 10^{-4}$. The computational results are presented in the following table.

Example	(m,n)	TN error	Iterations
1	2,1	2.5676e-016	2
2	6,3	1.4775e-016	3
3	10,6	4.5119e-014	3
4	14,10	8.7338e-016	4
5	20,14	1.2256e-014	8
6	23,18	1.1109e-013	12
7	30,19	5.6293e-013	15

Table 1: TN errors for STLN solutions and Iterations at $\mu = 10^{-4}$,m, n = degree of f, g respectively.

Here in table 1, m, n denote the degree of polynomials f and g respectively. As shown in this table, the STLN method converges quickly to the optimal solutions. For example, in the third test where the $(m, n) = (10, 6)$, only 3 iterations needed to achieve solution with about e^{-14}error.

All the solutions appears in table 1, have been obtained with acceptable amount of error which did not exceed e^{-13}in all cases. Also, it did not require more than 15 iterations each time.

Based on the previous computational results and the criteria mentioned in (5.2.1) , it appears that STLN is very robust and efficient to calculate an optimal solution with the minimum error. It is proven the STLN suitability to get accurate answer for the ratio f/g in a polynomial form in a reasonable time. This suitability has been tested using different cases of polynomials roots taking into account the degree of the noise in the input data.

5.3 Conclusion

This chapter was the core of the whole work. It carried out number of test examples that were used to evaluate the proposed method. It investigated and critically evaluated using different criteria.

The completed experiments were explained clarifying the aim and the input of each experiment. The computational results that are obtained clearly shown and presented in a suitable format.

A critical evaluation of the results is given. As a result of the previous computational results, it is shown to what extent the STLN method is powerful and sufficient. The performance is demonstrated in different cases of polynomials where the coefficients are widely diverse.

Furthermore, the method has maintained its efficiency even if the amount of the noise is massive. It is shown its superiority when it is compared with the simple least squares solutions.

To sum up, the method is powerful tool to be used later in polynomials computations with different matrix structures.The performance might be enhanced when the input data is preprocessed.

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6 Conclusion and future work

This dissertation has considered one important application of the structured matrix methods in computer algebra. In particular, the purpose of the dissertation was to investigate the use of the STLN method on the Toeplitz matrix to address the problem of two divisible polynomials deconvolution when the noise is added.

The procedure of addressing this issue has been performed in three main stages. Formulating the inexact polynomials deconvolution into least squares equality (LSE) approach was the first stage. It involved the use of the Toeplitz matrix to represent the input data. The second stages involved applying an iterative algorithm based on the STLN method using the Toeplitz matrix structure in order to find the exact solution in polynomial form. It implied the use of some mathematical techniques including the QR factorization.

After developing the MATLAB program, a number of experiments have been carried out. Each experiment focused on specified test cases of the input data.The computational results have been evaluated and assessed based on specific criteria.

As a result, it has been shown the efficiency and the accuracy of the STLN method in finding an exact solution of f/g when g is exact divisor of f in the presence of noise.

Finally, the STLN method needs more investigation on a range of real life applications in different areas that require further study. The results can be enhanced when the input polynomials preprocessed using specific concepts. Further study can be done in case of multivariate polynomials using different structured matrix.

Abstract

The problem of inexact polynomials deconvolution arises in several applications, including signal processing and system identification. This issue can be investigated in order to find its solution, taking advantages of the interrelationship between structured matrices and polynomials computations. A solution can be obtained by first translating it into a structured total least squares approach of least squares problems. This translation allows structured matrix methods to be used on the input data matrix. The focus of this dissertation is on how to find an exact solution for the deconvolution of two inexact polynomials, using a structured matrix method. In particular, the STLN method on a Toeplitz matrix structure is implemented in a MATLAB program. The efficiency of such method has been proven after testing the produced software.

Keywords: total least square problem, Toeplitz matrix, deconvolution, STLN, inexact polynomials.

Abbreviations

$\mathbb{R}$	Real number set
m ,n	degree of polynomials f,g
T(g)	Toeplitz matrix of vector g
STLS	Structured Total Least Squares
STLN	Structured Total Least Norm
LS	Least Squares
LSE	Least Squares Equality
TN	Total Norm
μ	The noise to noise ratio.

Contents

The University Of Sheffield.

The Applications of Structured Matrix methods

By

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This report is submitted in partial fulfillment of the requirement for the degree of MSc in Advance Software Engineering.

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Abstract

The problem of inexact polynomials deconvolution arises in several applications, including signal processing and system identification. This issue can be investigated in order to find its solution, taking advantages of the interrelationship between structured matrices and polynomials computations. A solution can be obtained by first translating it into a structured total least squares approach of least squares problems. This translation allows structured matrix methods to be used on the input data matrix. The focus of this dissertation is on how to find an exact solution for the deconvolution of two inexact polynomials, using a structured matrix method. In particular, the STLN method on a Toeplitz matrix structure is implemented in a MATLAB program. The efficiency of such method has been proven after testing the produced software.

Keywords: total least square problem, Toeplitz matrix, deconvolution, STLN, inexact polynomials.

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Abbreviations

$\mathbb{R}$	Real number set
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Contents

1 Introduction

In applied science and engineering, polynomials computation has an essential role, since many problems can be represented as operations on polynomials. While some of the basic operations on polynomials seem straightforward and their solutions can be obtained easily, polynomials division (deconvolution) is one of the computational problems. Even if one polynomial is an exact divisor of the other, the result would not be in a polynomial form.

In practical applications, this problem becomes more obvious where the coefficients of the input polynomials are usually expected to have a degree of noise. This noise is due to the rounding off in the previous computations. As a result, the input polynomials being turned into inexact polynomials. Therefore, deconvolving such polynomials will most likely be non trivial.

Over past decades, number of studies in computer algebra have been carried out to investigate such problem. Consequently, the interrelationship between fundamental computations with polynomials and rational functions and computations with structured matrices has been demonstrated. Moreover, it has been shown that structured matrices works as a bridge between computations with polynomials and numerical matrix computations.

Various proposed methods on structured matrices involving deep mathematical tools were found to solve several problems in these fields. In that sense, to work more sensibly with inexact polynomials, each polynomial problem should be translated into the terms of a structured matrix.

This project focuses on solving the problem of two inexact polynomials deconvolution when one polynomial is an exact divisor of the other. It will benefit from the application of structured matrices in polynomials computations. It will investigate the use of the Toeplitz matrix as a structure matrix to represent the input data and formulate the problem of deconvolution as a structured total least squares problem.

1.1 Aim

This project aims to consider the application of a structure preserving method to address the problem of deconvolving two divisible polynomials, when the noise is present in either one polynomial or in both. The following example illustrates the main goal of this work.

Consider the exact polynomials $f(x)$ and $g(x)$ where g is an exact divisor of f:

$$f(x) = (x-1)(x+3) \text{ and } g(x) = (x-1),$$

therefore f/g can be computed easily which is $f/g = (x+3)$.

However, in reality where the coefficients of f and g are affected by the noise which makes:

$$\widehat{f} = f(x) + \triangle f = (x - 1.02)(x + 3.01)$$

$$\text{and } \widehat{g} = g(x) + \triangle g = (x - 1.002),$$

Thus, the result of $\hat{f}/\hat{g}$ will be a rational function.
According to the structured total least norm (STLN) method technique, correcting the noise with the minimal perturbation on $f(x)$ and $g(x)$ leads to a solvable equation. In that sense, the work will focus on computing the solution h of:

$$h = \tfrac{\widehat{f}+s}{\hat{g}+z},$$

in a polynomial form satisfying the constrain that s and z are in the possible minimization. The strategy of the work is mainly divided into three stages.The first stage is to formulate the polynomials deconvolution into a structured matrix-vector multiplication using a Toeplitz matrix in particular.

Then, in the next stage, the structured total least norm (STLN) method will be applied to solve the resulting least square equality problem. After that, the method will be implemented in a MATLAB program and number of experiments will be conducted to investigate to what extent the (STLN) method will overcome the problem.

1.2 Dissertation Structure

The remaining part of this dissertation is structured as follows:

- Chapter 2: Mathematical Background: This chapter introduces clear descriptions and definitions for some basic mathematical concepts on polynomials and matrices.

- Chapter 3: Literature Review: This chapter presents the proposed approaches in the previous studies to solve the least squares problem. It critically evaluates each approach and outlines the nominated method that is used in the next chapter. In addition, it outlines and concludes some results of previous works.

- Chapter 4: Methodology: This chapter analyzes the main problem of inexact polynomials deconvolution. It gives a clear description on how it can be formulated into a least squares equality (LSE) problem. It summarizes the STLN based algorithm that will be implemented

•Chapter 5: Results and Discussion: This chapter evaluates the proposed algorithm. It presents the results of a number of experiments that have been carried out on a developed MATLAB program. It critically evaluates the method performance and the accuracy of the computational results.

- Chapter 6: Conclusion and Future Work: This chapter concludes the final work with some suggestions for future work.

2 Mathematical Background

2.1 Introduction

This chapter presents some basic mathematical concepts in polynomials that are necessary to declare and used in the later chapters. In addition to some matrices concepts since each polynomials operation can be translated into a matrix system.

This translation helps to address the ill posed operations in polynomials computations such as polynomials deconvolution which can be turned and formulated into least squares problem.

The chapter firstly, defines general polynomials aspects in the first section then the followed section focuses on the structured matrices. Then, mathematical operations on polynomials such as convolution and deconvolution are explained in section 2.4. The remainder of the chapter illustrates the way of using matrices to solve linear system equations.

2.2 Polynomials

2.2.1 Approximate (inexact) polynomials

In real applications, the polynomial's coefficients have some error added to them. Thus, the polynomial becomes inexact polynomial or approximate polynomial. In practical :

$$\hat{f} = f + \triangle f. \tag{2.1}$$

2.2.2 Polynomial coefficients norms

For each polynomial: $f(x) = \sum_{k=0}^{n} a_k x^k$ and $a = \begin{bmatrix} a_0 & a_1 & \ldots & a_n \end{bmatrix}$ is a vector of its coefficients, it has several classes of norms, which can be defined as:

$$||a||_p = (\sum_{k=0}^{n} |a_i|^p)^{1/p}$$

In this report the $||a||$ will be used as a standard for $||a||_2$.

2.3 Structured Matrix

2.3.1 Toeplitz Matrix

In mathematics, a Toeplitz is a matrix in which each descending diagonal from left to right is constant. The Toeplitz matrix from a given vector $a = \begin{bmatrix} a_0 & a_1 & a_2 \end{bmatrix}$:

$$T(a) = \begin{bmatrix} a_0 & & \\ a_1 & a_0 & \\ a_2 & a_1 & a_0 \\ & a_2 & a_1 \\ & & a_2 \end{bmatrix}$$

2.4 Convolution and Deconvolution

2.4.1 Convolution

Convolution is a mathematical operation that takes two functions f and g to produce a third function h which is a modified version of the two input functions. Algebraically, convolution is equivalent to polynomial multiplications.

2.4.2 Deconvolution

It is a non-trivial operation which is equivalent to polynomials division. If the ratio $f(x)/g(x)$ is a polynomial, a random noise in $f(x)$ and/or $g(x)$ makes it a rational function. Therefore the deconvolution of two inexact polynomials is an ill-posed problem.

2.5 Representing Linear Algebraic Equations in Matrices

It has been shown that matrices provide a concise notation for representing and solving linear equations. For example:

$$\begin{cases} a_1x_1 + b_1x_2 + c_1x_3 = d_1 \\ a_2x_1 + b_2x_2 + c_2x_3 = d_2 \\ a_3x_1 + b_3x_2 + c_3x_3 = d_3 \end{cases} \tag{2.2}$$

The above (1) system of equations can be represented and solved by matrices as follows:

$$Ax = b \tag{2.3}$$

Where A is the matrix of coefficients, b is the column vector of constants and x is the column vector of unknowns :

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$b^T = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}$

$x^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$

Then to solve equation (2):

$x = A^{-1}b$

Therefore, the value of x has been determined. In this work, we will use the Moore–Penrose pseudo inverse.

2.6 Summary

In conclusion, dealing with operations on polynomials requires dealing with matrices. The chapter declared the necessary mathematical concepts and notations related to polynomials together with matrices. A clear explanation of how to represent equations of polynomials as operation on matrices has been provided in the last section of this chapter.

3 Literature review

3.1 Introduction

This chapter evaluates the different proposed methods taken to address the structured total least squares approach problem. The chapter starts by explaining the least squares problem, discussing how the problem can be extended to the different approaches.

Then, it focuses deeply on the structured Total Least Squares approach, outlining some methods developed to address such problems. Many problems in various areas can be formulated as a structured Total Least Squares problem, for example, system identification, computer algebra, and speech and sound processing. The chapter concludes the computational results of the previous works that have been done an published.

3.2 Least squares problems

The Least Squares problem (LS) involves finding x that satisfies the following minimization:

$$min_x ||Ax - b|| \tag{3.1}$$

in order to solve an over determined linear equation: $Ax \approx b$.Many proposed methods to solve such problems allow the noise to be added into vector b only and assuming that the given matrix A is known without error.

However, by allowing the possibility of error in the elements of data matrix A ,we can obtain more accurate solutions for the entire equations. This extension for the LS problem known as total least square (TLS problem)can be stated as finding the vector x with the minimization:

$$min_{||\triangle A||,||\triangle b||} ||A + \triangle A)x - (b + \triangle b)|| \tag{3.2}$$

Golub and Van Loan(1980) introduced the basic least squares problem (3.1) along with the solution via singular value decomposition in their paper. Years later, this solution had been generalized to overcome the problem of multivariable and non generic cases.

Finally, the extension of TLS is the structure total least squares (STLS) which satisfies the same minimization needed in TLS while preserving the structure of matrix A. This additional constraint plays an important role in many computer science applications such as signal processing, system identification and system response prediction. A solution to this problem will allow us to form the output polynomials without the concerns of matrix A structure.

De Moor (1994) has mentioned various applications of the structured total least squares problem, with an exception of the numerical solution with the help of a Newton-type optimization method on the constrained total least squares problem.

He has also provided an outline regarding a new framework that would be helpful in deriving numerical methods and analytical properties. He has based his approach on the Lagrange multiplier, which yields equivalent problem termed as Reimannian singular value decomposition.

3.3 Structured Total Least Squares problem

To date, various methods have been developed and introduced to solve STLS problems. Van Huffel & Lemmerling (2002) described three different approaches, which are the Constrained Total Least Square algorithm (CTLS) proposed by Abatzoglou and Mendel in 1987, the Riemannian Singular Value Decomposition (RiSVD) algorithm by De Moor (1993, 1994) and the Structure Total Least Norm (STLN) algorithm by Van Huffel et al. (1996).

While all of these approaches were developed to satisfy the minimization in 3.2, each approach has its own field of applications. The most significant difference that distinguishes the last method (STLN) and makes it straightforward, is that it basically starts from the exact formula of the problem, while the others derive an equivalent formulation for which each algorithm, then develop it in a quite different series of steps.

Lemmerling et al. (1996) declared CTLS approach with the number of different methods that have been used. This paper has proven the convergence of the value obtained by using such an approach with the value obtained in solving TLS problem with other approaches.

However, the convergence rate depends totally on which method has been selected to solve the CTLS approach (Lemmerling et al., 1996). Abatzoglou et al. (1991) identified several advantages of applying the CTLS technique in Harmonic super-resolution problems. Indeed, the CTLS is a useful tool in signal processing problems where the known error present in the data matrix is algebraically associated, and there must be a solution for that equation.

Besides these contributions, a lot of research still needs to be performed into how to improve the performance of that approach.

According to De moor (1994), the STLS problem is equivalent to non-linear singular value decomposition. Then using this technique with one of the proposed methods will result in producing the solution for the STLS problem .

It is evident the efficiency of such an approach to solve the noisy realization problem. Furthermore, the study that has been done by (Fierro and Jiang ,2005) confirmed the reliability of the RiSVD approach in information retrieval applications. However, according to (Van Huffel et al.,1996) the result obtained by such an approach does not guarantee the preservation of the matrix structure, which is a requirement of the optimal solutions for STLS problems.

Rosen et al. (1996) proposed structured total least norm which is another algorithm for calculating structured total least squares solution. This method has been applied to solve a range of problems in various applications such as system identification, speech and audio processing and computer algebra.

The efficiency and robustness of this algorithm in addressing the structured total least squares problems have been proven theoretically and practically especially when the error can occur in the input data.

The choice of STLN approach is supported by the accurate results reported in many published papers and researches ;Winkler and Allan (2008) and (Van Huffel et al., 1996). STLN proved its efficient performance in many approximate polynomials and structured matrices applications where the STLS problems arises.

3.4 Overview of Structured Matrices Applications on polynomials

In light of the previous section, STLN has been used in many different applications. For example, noise realization, image reconstruction, system identification and signal processing. This section highlights some applications.

Winkler and Allan (2008) have developed in their work a method to compute the greatest common divisor (GCD) of two inexact polynomials. The main problem has been formulated into STLN as follow:

$$min||z|| \text{ with } (A_k + E_k)x = b_k + h_k$$

for some vector x,where the perturbation matrix $[h_k, E_k]$ has the same structure of the Sylvester matrix $[b_k, A_k]$ and z is the correction vector.Thus, they was aiming to find x that satisfied the minimum perturbation in both input polynomials.After the STLN method has been examined, an accurate results has been obtained from different test cases without needing to large number of iterations.

Apart of computer algebra, the STLN method also has been used in engineering applications such as signal processing, linear prediction and noise realization problems. More precisely, the Hankel structure has been used in noisy realization problem. According to (Moor B., 1994) for a given exact data $a \in \mathbb{R}^{p+q-1}$by the perturbed $b \in \mathbb{R}^{p+q-1}$, the noise can be realized by solving the following minimization:

$$\sum_{i=1}^{p+q-1}(a_i - b_i)^2 \text{ subject to } By = 0, y^t y = 1,$$

where B is $p \times q$ Hankel matrix constructed from the elements of b.

Markovsky and Huffel (2007) reviewed the total least squares methods in their paper in deep discussion and a comprehensive explanation.They mentioned different applications that used different structured matrices associated with their STLS formulas. Conclusion of their work and the above, it is evident that STLS solutions were the optimal solutions for many problems when its structured appropriately.

3.5 Summary

A considerable amount of literature in least squares problem has been researched. Least squares problem can be extended into total least squares(TLS) and structured total least squares (STLS) approaches depending on the limitation of the problem.

Each approach has its own applications and constrains. In computer algebra, polynomials computations can be turned into STLS problem.

Structured matrices have been studied closely for a long time in somewhat different fields, such as mathematics, computer science and engineering. Number of papers that summarized these studies of the structured matrices have been reviewed in this work.

In summary, STLN that is written by (Rosen et al. ,1996) is the most appropriate method to be used in the deconvolution problem. It will be adapted using appropriate structure (Toeplitz structure). A full description of the proposed method considered in the following chapter.

4 Methodology

4.1 Introduction

This chapter explains the first two stages that have been clarified in the first chapter of this dissertation. It illustrates the problem of deconvolving two inexact polynomials and how it can be formalized into a least square problem.
Then, it provides a suggested method to construct a Toeplitz matrix that will be used to represent the input polynomials coefficients. Next, it describes the method of structured total least norm(STLN) for the solution of the deconvolution problem. Furthermore, it outlines some techniques that are used while developing the MATLAB program.

4.2 Toeplitz matrix-vector Multiplication

Suppose that we have two polynomials $f(x)$ and $g(x)$ of degrees m and n respectively, as follow :

$$f(x) = \sum_{i=0}^{m} a_i x^{m-i} \text{ and } g(x) = \sum_{i=0}^{n} b_i x^{n-i}$$

and thus the polynomial

$$h(x) = f(x)/g(x) \tag{4.1}$$

is of degree $(m-n)$,

$$h(x) = \sum_{i=0}^{m-n} h_i x^{m-n} \tag{4.2}$$

Then (4.1) can be written in a matrix-vector multiplication form using a Toeplitz structure as

$$T(g)h = f \tag{4.3}$$

where $T(g) \in \mathbb{R}^{(m+1)\times(m-n+1)}$, $h \in \mathbb{R}^{(m-n+1)}$ and $f \in \mathbb{R}^{(m+1)}$ are coefficients vectors of polynomials $h(x)$ and $f(x)$respectively.

$$T(g) = \begin{bmatrix} b_0 & & & \\ b_1 & b_0 & & \\ \vdots & b_1 & \ddots & \\ & \vdots & \ddots & b_0 \\ b_n & & \ddots & b_1 \\ & b_n & & \vdots \\ & & & b_n \end{bmatrix}, \ h = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{m-n} \end{bmatrix} \text{ and } f = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

The deconvolution problem requires calculating h when f and g are given, which implies finding the least squares solution of (4.3).

4.3 Solving the Least Squares problem

4.3.1 The Moore–Penrose pseudoinverse

The simplest way to find the least squares solution is by using the pseudo inverse as followss:

$$h = T(g)^{\dagger} f, \tag{4.4}$$

This way will be used first to calculate the initial value of h.

4.3.2 The Structured Total Least Norm method

In order to apply the structure preserving method to find the solution for (4.3), it is required that the coefficients of of the input polynomials be perturbed slightly. Therefore, the coefficients of $h(x)$ can be calculated more accurately. Thus, the equation (4.3) will be written as follows

$$(T(g) + E(z))h = f + s, \tag{4.5}$$

where $E(z) \in \mathbb{R}^{(m+)\times(m-n+1)}$ has the same structure of $T(g)$, and the vector $s \in \mathbb{R}^{(m+1)}$ is the correction vector for the polynomial f.

$$E(z) = \begin{bmatrix} z_0 & & & \\ z_1 & z_0 & & \\ \vdots & z_1 & \ddots & \\ & \vdots & \ddots & z_0 \\ z_n & & \ddots & z_1 \\ & z_n & & \vdots \\ & & & z_n \end{bmatrix} \text{ and } s = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{bmatrix}.$$

Thus, the vectors z and s need to be computed. Supposing that the residual r using an approximate solution of (4.5) is

$$r = r(s, z) = (f + s) - (T(g) + E(z))h, \tag{4.6}$$

then

$$\begin{aligned} r(s + \delta s, z + \delta z) &= (f + (s + \delta s)) - (T(g) + E(z + \delta z))(h + \delta h) \\ &= r(s, z) + \delta s - (T(g) + E(z))\delta h - (\delta(E(z))h, \end{aligned}$$

where

$$\delta E(z) = \sum_{i=0}^{n} \frac{\partial E}{\partial z_{zi}} \delta z_i. \tag{4.7}$$

There exists a matrix $Y(h) \in \mathbb{R}^{(m+1)\times(n+1)}$ that satisfies the following equation:

$$E(z)h = Y(h)z, \tag{4.8}$$

So, we can substitute $(\delta(E(z))h$ in $r(s + \delta s, z + \delta z)$ equation with $Y(h)\delta z$, That leads to change r to:

$$\begin{aligned} r(s + \delta s, z + \delta z) &= (f + (s + \delta s)) - (T(g) + E(z + \delta z))(h + \delta h) \\ &= r(s, z) + \delta s - (T(g) + E(z))\delta h - (Y(h))\delta z, \end{aligned}$$

In order to solve (4.3) using the Newton-Raphson methods, it implies an iterative solution for the residual

$$\begin{bmatrix} Y & (T + E) & I_{m+1} \end{bmatrix} \begin{bmatrix} \delta z \\ \delta h \\ \delta s \end{bmatrix} = r, \tag{4.9}$$

Hence, it is required to follow the minimization of

$$|| \begin{bmatrix} \delta z & \delta h & \delta s \end{bmatrix} ||, \tag{4.10}$$

Subject to

$$\begin{bmatrix} Y & (T+E) & I_{m+1} \end{bmatrix} \begin{bmatrix} \delta z \\ \delta h \\ \delta s \end{bmatrix} = r, \tag{4.11}$$

That leads to least squares equality problem, which can be solved by the QR decomposition technique. Suppose that

$$F = I_{2m+3,}, \qquad G = \begin{bmatrix} Y & (T+E) & I_{m+1} \end{bmatrix}, \qquad y = \begin{bmatrix} \delta z \\ \delta h \\ \delta s \end{bmatrix},$$

$$S = \begin{bmatrix} -z_i \\ -(h_i - h_0) \\ -s_i \end{bmatrix} \text{ and } t = r_i,$$

Therefore, this works considers the following LSE problem:

$min_y ||Fy - S||$ Subject to $Gy = t$.

That means we need to overcome the noise that turn the polynomial f/g to a rational function. STLN method would correct the noise with minimizing the perturbation as much as possible.

The following algorithm is generated base on STLN. Since the input of the algorithm is inexact polynomials, a random noise with ratio μ will be added to f and g firstly. The stop condition for the iterative method is when the total norm error in the computed solution $\leq 10^{-12}$ or after 100 iterations. The reason for choosing this number is that no improvement will be noticed when the TN error becomes less than 10^{-12}.

The method denotes to the corrections added to f and g with z and s respectively. The values of these correction vectors initialized by zeros. Taking into account that z has the same structure of $T(g)$. As, it is clear in the algorithm, powerful mathematical techniques will be used such as QR factorization.

Algorithm 1 Deconvolution using QR decomposition.

Input: Inexact polynomials $f(x)$ and $g(x)$.
Output: The polynomial $h(x) = f(x)/g(x)$.
Begin

1. Set $z_0 = 0$, $s_0 = 0$ and calculate $h_0$from (4.4)
2. Repeat

- Compute the QR decomposition of G^T,

$$G^T = QR = Q \begin{bmatrix} R_1 \\ 0 \end{bmatrix}. \tag{4.12}$$

- Set $w_1 = R_1^{-T} \in \mathbb{R}^{(m-n+1)}$.
- Partition FQ as

$$FQ = \begin{bmatrix} F_1 & F_2 \end{bmatrix}, \tag{4.13}$$

  where $F_1 \in \mathbb{R}^{(2m+3)\times(m+1)}$ and $F_2 \in \mathbb{R}^{(2m+3)\times(m+2)}$.
- Compute

$$w_2 = F_2^{\dagger}(S - F_1 w_1) \in \mathbb{R}^{(m+2)}. \tag{4.14}$$

- Compute the solution

$$y = Q \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{4.15}$$

- Set $z := z + \delta z$, $h := h + \delta h$ and $s := s + \delta s$.
- Update $E(z)$ and $Y(h)$.
- Update G, S and t, then compute the residual $r(z)$ from (4.7).

Until$\frac{||r(z)||}{||f+s||} \leq 10^{-12}$.

End

4.3.3 Highlight on MATLAB code

Here are functions that are used while developing the MATLAB program. The program takes the input polynomials in roots form. So, it is necessary to generate the polynomial using the roots before starting the iterative method. The following function is written to accomplish this task.

```
function [p]=CreatePolynomial(a)
[row col]=size(a);
p=1;
for i=1:row
    C=[ 1 -(a(i,1))];
     for j=1:a(i,2)
         p=conv(p,C);
     end
end
end
```

Moreover, the Toeplitz matrix of the vector g, should be constructed properly. The function Toeplitz(g,m) is developed to achieve this structure.

```
function T = Toeplitz(g,m)
n=length(g)-1; % the degree of g
T=zeros(m+1,m-n+1);
for k=1:1:m-n+1
     for l=k:1:n+k
        T(l,k)=g(l-k+1);
     end
end
end
```

In order to create the matrix $Y(h)$ using the vector h which satisfies $E(z)h = Y(h)z$, the following MATLB function has been written.

```
function Y=createY(h,m,n)
Y=zeros(m+1,n+1);
for k=1:1:n+1
    for l=k:1:m-n+k
        Y(l,k)=h(l-k+1);
    end
end
end
```

4.4 Summary

In conclusion, the polynomials deconvolution problem has been analyzed and how it can be transformed into convolution with a Toeplitz matrix form has been clarified.

It has been shown how it leads to least squares equality (LSE) problem. An exact solution can be obtained with minimum perturbation by the (STLN) based algorithm which is explained in this chapter.

The proposed algorithm has been implemented in a MATLAB program. In order to validate the method in calculating an exact solution for the deconvolution problem, a series of experiments on the developed software are carried out.

The computational results are summarized and evaluated in the remaining parts of the dissertation.

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